On Broadcasting Time*

Vassilios V. Dimakopoulos *Nikitas J. Dimopoulos*

Technical Report ECE-95-5

Dept. of Electrical and Computer Engineering, University of Victoria
P.O. Box 3055, Victoria, B.C., CANADA, V8W 3P6.
Tel: (604) 721-8773, 721-8902, *Fax:* (604) 721-6052,
E-mail: {dimako, nikitas}@ece.uvic.ca

August, 1995

Abstract

Broadcasting is an information dissemination problem in which a particular node in a network must transmit an item of information to all the other nodes in the network. In this work we present new lower bounds for the time needed to complete this process in arbitrary graphs. In particular we generalize a result of P. Fraigniaud and E. Lazard [*Discrete Applied Mathematics*, **53** (1994) 79–133] which states that if in a graph there are at least two vertices in distance equal to the diameter from the originator, then broadcasting requires time at least equal to the diameter plus one.

Keywords: broadcasting, combinatorial problems, distributed computing, networks.

*This research was supported in part though grants from NSERC, IRIS and the University of Victoria.

1. Introduction

Information dissemination, apart from its theoretical interest, has always been of practical importance especially when it comes to network computing and, more recently, loosely-coupled multiprocessors. Broadcasting in particular is an essential operation in the majority of parallel algorithms. One important class is parallel numerical algorithms [3, 5].

Consider a connected graph (or *network*) G and a particular vertex (or *node*) v which we will call *source* or *root*. The model we will follow is the *single-port* one whereby each node may utilize only one of its incident links at a time. More formally, broadcasting from v involves the transmission of a message from v to the rest of the vertices in G subject to the following constraints:

- each transmission involves exactly two vertices
- each transmission requires one time unit (or *step*)
- a vertex may transmit to at most one other vertex per time unit
- a vertex may only transmit to an adjacent vertex.

The broadcasting time of a root vertex v, denoted $b(v)$, is the minimum number of time units required to complete broadcasting i.e. to have all other vertices informed about the message. The broadcasting time of G, $B(G)$, is the maximum broadcasting time of any vertex in G. One of the first known results is that since the number of informed vertices can at most double after each step, $B(G) \geq \log_2 N$, where N is the number of vertices in G.

There has been a substantial amount of work on the problem. Hedetniemi, Hedetniemi and Liestman [4] and Fraigniaud and Lazard [2] are two excellent surveys on the subject. A large portion of the work has been concentrated on constructing efficient broadcast graphs, i.e. graphs that allow for broadcasting in the minimum possible time and contain as few edges as possible; see for example [1] and the references therein. The rest of the work on the problem concentrates mostly on developing minimum-time algorithms or determining bounds on the broadcasting time of particular classes of graphs.

In this paper we derive general lower bounds for the broadcasting problem in arbitrary graphs. Except for $B(G) \geq \log_2 N$, the only other general bound known is that $B(G) \geq K$ where K is the diameter of the network; this is easy to see since nodes in distance K away from a source node cannot be reached in less than K steps. Fraigniaud and Lazard [2] showed that if there exist at least 2 nodes in distance K from the source node, then at least $K+1$ steps are required. We show here that a general formula may be derived for the case where the number of nodes in distance greater than a given constant is known.

2. A general lower bound

Consider a graph G and let v be the source node. We will assume that there exist $R_v(D)$ nodes in G which are in distance *at least* D from v. We are going to derive a lower bound for the broadcasting time, $b(v)$, of node v. Let $r(n,t)$ be the maximum number of nodes that can be reached through paths of length n from v and which get informed exactly at time t. Such nodes receive the message from nodes lying on a path of length $n-1$ from v who got informed some time before t. Therefore, $r(n,t)$ is described by the following recursion

$$r(n,t) = r(n-1,t-1) + r(n-1,t-2) + \cdots + r(n-1,n-1). \tag{1}$$

Notice that $r(n,t) = 0$ for $t < n$ and this is why the above recursion stops at the term $r(n-1,n-1)$. That is, the recursion holds for any $t \geq n$ and $n \geq 1$. Since only one node in distance 1 from the root node can be informed at any step $t \geq 1$, we obtain the boundary condition $r(1,t) = 1$. We finally define $r(0,0) = 1$; the source is assumed to become aware of the message at time 0. It should be noted that a limited-history version of the above recursion was derived in [6] for graphs with maximum degree of four.

Lemma 1

$$r(n,t) = \binom{t-1}{n-1}.$$

Proof. Eq. (1) can be written in a more familiar form by observing that $r(n-1,t-2) + r(n-1,t-3) + \cdots + r(n-1,n-1) = r(n,t-1)$. Consequently we have

$$r(n,t) = r(n-1,t-1) + r(n,t-1),$$

with $r(1,t) = 1$ for $t \geq 1$. Using the standard technique of generating functions [7], it can easily be shown that the solution of the above recursion is $r(n,t) = \binom{t-1}{n-1}$. ■

2.1. General networks

Based on the above, we may express a general lower bound on broadcasting from node v. Nodes that receive the message through paths of length n are not necessarily in distance n from v. In general, if a vertex is in distance n' from v, it may be informed through any path of length $n \geq n'$. Concentrate on distance D and assume that broadcasting

finishes at time $b(v) = T$. Then by time T all nodes in distance D or more must have been informed. Since we have assumed that there are $R_v(D)$ such nodes, we must have

$$\sum_{n=D}^{T} \sum_{t=n}^{T} r(n,t) \geq R_v(D).$$

From Lemma 1 we obtain

$$\sum_{n=D}^{T} \sum_{t=n}^{T} r(n,t) = \sum_{n=D}^{T} \sum_{t=n}^{T} \binom{t-1}{n-1}.$$

The sum on the right-hand side evaluates to $\binom{T}{n}$ [7]. Hence,

$$\sum_{n=D}^{T} \sum_{t=n}^{T} r(n,t) = \sum_{n=D}^{T} \binom{T}{n} \geq R_v(D). \tag{2}$$

We have thus proven the following:

Theorem 2 *If T is the minimum t such that $\sum_{n=D}^{t} \binom{t}{n} \geq R_v(D)$ then $b(v) \geq T$, i.e. broadcasting from v requires at least T steps.*

Notice that if $D = 0$, $R_v(D) = N$ where N is the number of nodes in the network. In this case, (2) reduces to $\sum_{n=0}^{T} \binom{T}{n} = 2^T \geq N$ which gives $T \geq \log_2 N$ — the well-known bound for any network.

We next set $D = K$, the diameter of the network. If $T = K$ then (2) shows that $R_v(K) \leq 1$ which was shown in [2]. Similar results can be obtained for other values of $T \geq K$. For example, if $T = K + 1$ then (2) gives $R_v(K) \leq K + 1$; consequently, if there exist more than $K + 1$ nodes in distance K from the source node, broadcasting requires at least $K + 2$ steps.

We next give an approximate formula for the minimum T due to the difficulty associated with handling inequality (2).

Corollary 3 *Broadcasting from v requires time at least equal to*

$$T = \begin{cases} 2D - 1 & \text{if } R_v(D) = 4^{D-1} \\ \log_2(4^{D-1} + R_v(D)) & \text{if } R_v(D) > 4^{D-1}. \\ \dfrac{D}{2} + [D!(R_v(D) - 1)]^{1/(D+1)} & \text{if } R_v(D) < 4^{D-1} \end{cases}$$

Proof. Setting $T = 2D - 1$, (2) evaluates to

$$\sum_{n=D}^{2D-1} \binom{2D-1}{n} = \frac{1}{2} 2^{2D-1} = 4^{D-1} \geq R_v(D),$$

hence the first branch of the result.

If $R_v(D) > 4^{D-1}$ then $T > 2D - 1$. We obtain

$$\begin{aligned} \sum_{n=D}^{T} \binom{T}{n} &= \sum_{n=0}^{T} \binom{T}{n} - \sum_{n=0}^{D-1} \binom{T}{n} \\ &\leq \sum_{n=0}^{T} \binom{T}{n} - \sum_{n=0}^{D-1} \binom{2D-1}{n} \\ &= 2^T - \frac{1}{2} 2^{2D-1}. \end{aligned}$$

Hence, $2^T \geq 4^{D-1} + R_v(D)$, or $T \geq \log_2(4^{D-1} + R_v(D))$.

If $R_v(D) < 4^{D-1}$ then $T < 2D - 1$. In this case the maximum term of the sum in (2) is $\binom{T}{D}$. Consequently,

$$\sum_{n=D}^{T} \binom{T}{n} \leq (T - D) \binom{T}{D} + 1.$$

By standard combinatorial properties, $(T - D)\binom{T}{D} = (D + 1)\binom{T}{D+1}$. Hence,

$$\sum_{n=D}^{T} \binom{T}{n} \leq (D + 1) \frac{(T - D)(T - D + 1) \cdots T}{(D + 1)!} + 1.$$

It is well known that given n numbers $a_1, \ldots, a_n$, their geometric mean $(a_1 \cdots a_n)^{1/n}$ is less or equal to their arithmetic mean $(a_1 + \cdots + a_n)/n$. Thus, we seek the minimum T such that

$$\frac{[T - \frac{D}{2}]^{D+1}}{D!} \geq R_v(D) - 1.$$

Taking the $(D + 1)$th root of both sides yields the desired result (last branch of the equation). As a final note, for large m, Stirling's approximation gives $m! \approx \sqrt{2\pi m}(m/e)^m$. Consequently, if D is large and $R_v(D) \leq 4^{D-1}$, the last branch of the corollary can be written as

$$T \geq \frac{D}{2} + \frac{D + 1}{e} \left(\frac{(R_v(D) - 1)\sqrt{2\pi}}{\sqrt{D + 1}} \right)^{\frac{1}{D+1}}$$

■

2.2. Trees

Trees are of particular interest since any non-redundant broadcasting algorithm (that is an algorithm that no vertex receives the message more than once) actually defines a spanning tree of the underlying network. The interested reader is referred to [8]. If the network is a tree, the path between any two nodes is unique and as a consequence, nodes informed through a path of length n are in distance exactly n from the root; (2) may thus take the simpler form

$$\binom{T}{D} \geq E_v(D),$$

where $E_v(D)$ is the number of nodes in distance *exactly* D from the root node.

Corollary 4 *If the network is a tree and T is the minimum t such that $\binom{t}{D} \geq E_v(D)$ then $b(v) \geq T$, i.e. broadcasting from v requires at least T steps.*

It is interesting to note that if v is the root and K is the height of the tree, then according to the above corollary, if $T = K$ there must exist at most $\binom{K}{L}$ nodes at any level L of the tree. Trees with exactly $\binom{K}{L}$ nodes at every level $L = 0, 1, \ldots, K$ and which achieve this minimum broadcasting time are unique and are known as *binomial* trees [8].

3. Summary

We derived lower bounds on the time needed to broadcast from a vertex in arbitrary networks. The bounds may be viewed as a generalization of a result in [2]. The formulas utilize information on the distance distribution of the source node and the known bounds of $B(G) \geq \log_2 N$ and $B(G) \geq K$ become special cases.

Tighter bounds can be derived if it is known that the network if of bounded degree, by considering a recursion similar to (1) but with limited history. The case of maximum degree of four was considered in [6] while general bounds based only on N were given in [1].

References

[1] J. - C. Bermond, P. Hell, A. L. Liestman and J. G. Peters, "Broadcasting in bounded degree graphs," *SIAM J. Discrete Math.*, Vol. 5, No. 1, pp. 10–24, Feb. 1992.

[2] P. Fraigniaud and E. Lazard, "Methods and problems of communication in usual networks," *Discrete Appl. Math.*, Vol. 53, pp. 79–133, 1994.

[3] D. B. Gannon and J. van Rosendale, "On the impact of communication complexity on the design of parallel numerical algorithms," *IEEE Trans. Comput.*, Vol. C-33, No. 12, pp. 1180–1194, Dec. 1984.

[4] S. M. Hedetniemi, S. T. Hedetniemi and A. L. Liestman, "A survey of gossiping and broadcasting in communication networks," *Networks*, Vol. 18, pp. 319–349, 1988.

[5] S. L. Johnsson, "Communication efficient basic linear algebra computations on hypercube architectures," *J. Parallel Distrib. Comput.*, Vol. 4, pp. 133–172, 1987.

[6] R. Klasing, B. Monien, R. Peine and E. A. Stöhr, "Broadcasting in butterfly and deBruijn networks," *Discrete Appl. Math.*, Vol. 53, pp. 183–197, 1994.

[7] C. L. Liu, *Introduction to combinatorial mathematics*. New York: McGraw-Hill, 1968.

[8] A. Proskurowski, "Minimum broadcast trees," *IEEE Trans. Comput.*, Vol. C-30, No. 5, pp. 363–366, May 1981.